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# 2-local isometries on spaces of differentiable functions

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Abstract

Let $C^{(2)}([0,1])$ be the Banach space of 2-times continuously differentiable functions on the closed unit interval $[0,1]$ equipped with the norm $\|f\|_\sigma = |f(0)| + |f'(0)| + \|f''\|_\infty$, where $\|g\|_\infty = \sup\{|g(t)| : t \in [0,1]\}$ for $g$. If $T : (C^{(2)}([0,1]), \|\cdot\|_\sigma) \to (C^{(2)}([0,1]), \|\cdot\|_\sigma)$ is a 2-local isometry, then $T$ is a surjective complex-linear isometry.

## 1 Introduction

Let $(M, \|\cdot\|_M)$ and $(N, \|\cdot\|_N)$ be normed linear spaces over the complex number $\mathbb{C}$. A mapping $T : M \to N$ is called an *isometry* if $\|T(f) - T(g)\|_N = \|f - g\|_M$ for all $f, g \in M$. The linear isometries on various function spaces have been studied by many mathematicians (see [2]). The source of this subject is the classical Banach-Stone theorem, which characterizes the surjective complex-linear isometry on $C(X)$, the Banach space of all complex-valued continuous functions on a compact Hausdorff space $X$ with the supremum norm $\|\cdot\|_\infty$.

**Theorem 1.1** (Banach-Stone). *A mapping $T$ is a surjective complex-linear isometry on $C(X)$ if and only if there exist a unimodular continuous function $w : X \to \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and a homeomorphism $\varphi : X \to X$ such that $T(f) = w(f \circ \varphi)$ for all $f \in C(X)$.*

In this paper, we treat with the space of continuously differentiable functions. Let $C^{(n)}([0,1])$ be the Banach space of all $n$-times continuously differentiable functions on the closed unit interval $[0,1]$ with a norm. For example, $C^{(n)}([0,1])$ with one of the following norms is a Banach space;

$$\|f\|_C = \sup_{t\in[0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!},$$

$$\|f\|_\Sigma = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty}{k!},$$

$$\|f\|_\sigma = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \|f^{(n)}\|_\infty,$$

$$\|f\|_m = \max\{|f(0)|, |f'(0)|, \dots, |f^{(n-1)}(0)|, \|f^{(n)}\|_\infty\},$$

for $f \in C^{(n)}([0,1])$. Among them, $(C^{(n)}([0,1]), \|\cdot\|_C)$ and $(C^{(n)}([0,1]), \|\cdot\|_\Sigma)$ are unital semisimple commutative Banach algebras. In 1965, Cambern [1] characterized surjective complex-linear isometries on $(C^{(1)}([0,1]), \|\cdot\|_C)$. In 1981, Pathak [10] extended this result to $(C^{(n)}([0,1]), \|\cdot\|_C)$. On the other hand, Rao and Roy [11] gave the characterization of surjective complex-linear isometries on $(C^{(1)}([0,1]), \|\cdot\|_\Sigma)$ in 1971. Those results say that every surjective complex-linear isometry has the canonical form; $T(f) = w(f \circ \varphi)$. However, the author [6, 7] proved that surjective complex-linear isometries on $(C^{(n)}([0,1]), \|\cdot\|_\sigma)$ or $(C^{(n)}([0,1]), \|\cdot\|_m)$ have a different form.

In [9], Molnár introduced the notion of 2-local isometry as follows. For a Banach space $\mathcal{B}$, a mapping $T : \mathcal{B} \to \mathcal{B}$ is called a 2-*local isometry* if for each $f, g \in \mathcal{B}$ there exists a surjective complex-linear isometry $T_{f,g} : \mathcal{B} \to \mathcal{B}$ such that $T(f) = T_{f,g}(f)$ and $T(g) = T_{f,g}(g)$. Note that no surjectivity or linearity of $T$ is assumed. Molnár studied 2-local isometries on $B(H)$, the Banach algebra of all bounded linear operators on an infinite dimensional separable Hilbert space $H$. Let $C_0(X)$ be the Banach algebra of all complex-valued continuous functions on a locally compact Hausdorff space $X$ which vanish at infinity equipped with the supremum norm $\|\cdot\|_\infty$. For a first countable $\sigma$-compact Hausdorff space $X$, Győry [3] showed that every 2-local isometry on $C_0(X)$ is a surjective complex-linear isometry. Hosseini [4] studied generalized 2-local isometries on $(C^{(n)}([0,1]), \|\cdot\|_m)$. The authors, in [5, 8], considered 2-local isometries on the spaces $(C^{(n)}([0,1]), \|\cdot\|_C)$, $(C^{(1)}([0,1]), \|\cdot\|_\Sigma)$ and $(C^{(1)}([0,1]), \|\cdot\|_\sigma)$.

## 2 Results

The following theorem is the main result of this paper.

**Theorem 2.1.** *Every* 2*-local isometry on* $(C^{(2)}([0,1]), \|\cdot\|_\sigma)$ *is a surjective complex-linear isometry.*

The following characterization of surjective complex-linear isometries on $(C^{(2)}([0,1]), \|\cdot\|_\sigma)$ is important to the proof of the theorem. For any $f \in C([0,1])$, define $Sf \in C^{(1)}([0,1])$ by $(Sf)(t) = \int_0^t f(s)\,ds$ $(\forall t \in [0,1])$.

**Lemma 2.2** ([7]). *A mapping* $T$ *is a surjective complex-linear isometry on* $(C^{(2)}[0,1], \|\cdot\|_\sigma)$ *if and only if there exist unimodular constants* $\lambda, \mu \in \mathbb{T}$, *a unimodular continuous function* $w : [0,1] \to \mathbb{T}$ *and a homeomorphism* $\varphi : [0,1] \to [0,1]$ *such that one of the following holds:*
(i) $T(f)(t) = \lambda f(0) + \mu f'(0)t + (S^2(w(f'' \circ \varphi)))(t)$ $(\forall f \in C^{(2)}([0,1]), \forall t \in [0,1])$.
(ii) $T(f)(t) = \lambda f'(0) + \mu f(0)t + (S^2(w(f'' \circ \varphi)))(t)$ $(\forall f \in C^{(2)}([0,1]), \forall t \in [0,1])$.

From now on, we write simply $C^{(2)}$ for the Banach space $(C^{(2)}([0,1]), \|\cdot\|_\sigma)$. Let $T$ be a 2-local isometry on $C^{(2)}$. We define the map $U : C([0,1]) \to C([0,1])$ by $U(f) = (T(S^2 f))''$ for all $f \in C([0,1])$.

**Lemma 2.3.** *There exist a unimodular continuous function* $w : [0,1] \to \mathbb{T}$ *and a homeomorphism* $\varphi : [0,1] \to [0,1]$ *such that* $(T(f))'' = w(f'' \circ \varphi)$ *for all* $f \in C^{(2)}$.

*Proof.* Let $f, g \in C([0,1])$. Since $T$ is a 2-local isometry on $C^{(2)}$, there exists a surjective complex-linear isometry $T_{S^2f,S^2g}$ on $C^{(2)}$ such that $T(S^2 f) = T_{S^2f,S^2g}(S^2 f)$ and $T(S^2 g) = T_{S^2f,S^2g}(S^2 g)$. By Lemma 2.2, there exist a unimodular continuous function $w_{f,g} : [0,1] \to \mathbb{T}$ and a homeomorphism $\varphi_{f,g} : [0,1] \to [0,1]$ such that $(T_{S^2f,S^2g}(h))'' = w_{f,g}(h'' \circ \varphi_{f,g})$ for all $h \in C^{(2)}$. Define $U_{f,g}(h) = w_{f,g}(h \circ \varphi_{f,g})$ for all $h \in C([0,1])$. By the Banach-Stone theorem, we see that $U_{f,g}$ is a surjective complex-linear isometry on $C([0,1])$. We have

$$U(f) = (T(S^2 f))'' = (T_{S^2f,S^2g}(S^2 f))'' = w_{f,g}(f \circ \varphi_{f,g}) = U_{f,g}(f).$$

Similarly, $U(g) = U_{f,g}(g)$. Hence $U$ is a 2-local isometry on $C([0,1])$. By [3, Theorem 2], $U$ is a surjective complex-linear isometry on $C([0,1])$. Hence the Banach-Stone theorem implies that there exist a unimodular continuous function $w : [0,1] \to \mathbb{T}$ and a homeomorphism $\varphi : [0,1] \to [0,1]$ such that

$$U(f) = w(f \circ \varphi) \tag{2.1}$$

for all $f \in C([0,1])$.

Let $f \in C^{(2)}$. Put $g = S^2(f'')$. Since $T$ is a 2-local isometry on $C^{(2)}$, there exists a surjective complex-linear isometry $T_{f,g}$ on $C^{(2)}$ such that $T(f) = T_{f,g}(f)$ and $T(g) = T_{f,g}(g)$. By Lemma 2.2, there exist a unimodular continuous function $w_{f,g} : [0,1] \to \mathbb{T}$ and a homeomorphism $\varphi_{f,g} : [0,1] \to [0,1]$ such that $(T_{f,g}(h))'' = w_{f,g}(h'' \circ \varphi_{f,g})$ for all $h \in C^{(2)}$. Then we have

$$(T(f))'' = (T_{f,g}(f))'' = w_{f,g}(f'' \circ \varphi_{f,g}) = w_{f,g}(g'' \circ \varphi_{f,g}) = (T_{f,g}(g))'' = (T(g))'',$$

since $g'' = (S^2(f''))'' = f''$. Substituting $f = f''$ into (2.1), we have

$$(T(f))'' = (T(g))'' = (T(S^2(f'')))'' = U(f'') = w(f'' \circ \varphi).$$

Hence the lemma completes the proof. $\square$

We define the functions $\mathbf{1}$ and $\mathbf{id}$ by $\mathbf{1}(t) = 1$ $(\forall t \in [0,1])$ and $\mathbf{id}(t) = t$ $(\forall t \in [0,1])$, respectively.

**Lemma 2.4.** *There exist unimodular constants $\lambda, \mu \in \mathbb{T}$ such that one of the following holds:*
(i) $T(\mathbf{1}) = \lambda\mathbf{1}$ *and* $T(\mathbf{id}) = \mu\mathbf{id}$.
(ii) $T(\mathbf{1}) = \mu\mathbf{id}$ *and* $T(\mathbf{id}) = \lambda\mathbf{1}$.

*Proof.* Since $T$ is a 2-local isometry, there exists a surjective complex-linear isometry $T_{\mathbf{1},\mathbf{id}}$ on $C^{(2)}$ such that $T(\mathbf{1}) = T_{\mathbf{1},\mathbf{id}}(\mathbf{1})$ and $T(\mathbf{id}) = T_{\mathbf{1},\mathbf{id}}(\mathbf{id})$. By Lemma 2.2, there exist unimodular constants $\lambda, \mu \in \mathbb{T}$, a unimodular continuous function $w_{\mathbf{1},\mathbf{id}}$ and a homeomorphism $\varphi_{\mathbf{1},\mathbf{id}}$ such that one of the following holds:
(i) $T_{\mathbf{1},\mathbf{id}}(f)(t) = \lambda f(0) + \mu f'(0)t + (S^2(w_{\mathbf{1},\mathbf{id}}(f'' \circ \varphi_{\mathbf{1},\mathbf{id}})))(t)$ $(\forall f \in C^{(2)}, \forall t \in [0,1])$.
(ii) $T_{\mathbf{1},\mathbf{id}}(f)(t) = \lambda f'(0) + \mu f(0)t + (S^2(w_{\mathbf{1},\mathbf{id}}(f'' \circ \varphi_{\mathbf{1},\mathbf{id}})))(t)$ $(\forall f \in C^{(2)}, \forall t \in [0,1])$.
If (i) holds, then we have $T(\mathbf{1})(t) = T_{\mathbf{1},\mathbf{id}}(\mathbf{1})(t) = \lambda$ and $T(\mathbf{id})(t) = T_{\mathbf{1},\mathbf{id}}(\mathbf{id})(t) = \mu t$.
If (ii) holds, then we have $T(\mathbf{1})(t) = T_{\mathbf{1},\mathbf{id}}(\mathbf{1})(t) = \mu t$ and $T(\mathbf{id})(t) = T_{\mathbf{1},\mathbf{id}}(\mathbf{id})(t) = \lambda$.
Hence the lemma is proven. $\square$

**Lemma 2.5.** *One of the following holds:*
(a) $T(f)(0) = T(\mathbf{1})(0)f(0)$ $(\forall f \in C^{(2)})$ *and* $(Tf)'(0) = (T(\mathbf{id}))'(0)f'(0)$ $(\forall f \in C^{(2)})$.
(b) $T(f)(0) = T(\mathbf{id})(0)f'(0)$ $(\forall f \in C^{(2)})$ *and* $(Tf)'(0) = (T(\mathbf{1}))'(0)f(0)$ $(\forall f \in C^{(2)})$.

*Proof.* Let $f \in C^{(2)}$. Since $T$ is a 2-local isometry, there exist surjective complex-linear isometries $T_{\mathbf{1},f}$ and $T_{\mathbf{id},f}$ such that $T(f) = T_{\mathbf{1},f}(f) = T_{\mathbf{id},f}(f)$, $T(\mathbf{1}) =$

$T_{\mathbf{1},f}(\mathbf{1})$ and $T(\mathbf{id}) = T_{\mathbf{id},f}(\mathbf{id})$. By Lemma 2.2, there exist unimodular constants $\lambda_{\mathbf{1},f}, \mu_{\mathbf{1},f}, \lambda_{\mathbf{id},f}, \mu_{\mathbf{id},f} \in \mathbb{T}$ such that one of the following (i) and (ii) and one of the following (I) and (II) hold:

(i) $T_{\mathbf{1},f}(g)(0) = \lambda_{\mathbf{1},f} g(0)$, $(T_{\mathbf{1},f}(g))'(0) = \mu_{\mathbf{1},f} g'(0)$ for all $g \in C^{(2)}$.

(ii) $T_{\mathbf{1},f}(g)(0) = \lambda_{\mathbf{1},f} g'(0)$, $(T_{\mathbf{1},f}(g))'(0) = \mu_{\mathbf{1},f} g(0)$ for all $g \in C^{(2)}$.

(I) $T_{\mathbf{id},f}(g)(0) = \lambda_{\mathbf{id},f} g(0)$, $(T_{\mathbf{id},f}(g))'(0) = \mu_{\mathbf{id},f} g'(0)$ for all $g \in C^{(2)}$.

(II) $T_{\mathbf{id},f}(g)(0) = \lambda_{\mathbf{id},f} g'(0)$, $(T_{\mathbf{id},f}(g))'(0) = \mu_{\mathbf{id},f} g(0)$ for all $g \in C^{(2)}$.

If (i) and (I) hold, we have $T(f)(0) = T_{\mathbf{1},f}(f)(0) = \lambda_{\mathbf{1},f} f(0)$ and $(T(f))'(0) = (T_{\mathbf{id},f}(f))'(0) = \mu_{\mathbf{id},f} f'(0)$. Also, we have $T(\mathbf{1})(0) = T_{\mathbf{1},f}(\mathbf{1})(0) = \lambda_{\mathbf{1},f}$ and $(T(\mathbf{id}))'(0) = (T_{\mathbf{id},f}(\mathbf{id}))'(0) = \mu_{\mathbf{id},f}$. Hence we obtain (a).

If (i) and (II) hold, $T(\mathbf{1})(0) = T_{\mathbf{1},f}(\mathbf{1})(0) = \lambda_{\mathbf{1},f} \in \mathbb{T}$ and $T(\mathbf{id})(0) = T_{\mathbf{id},f}(0) = \lambda_{\mathbf{id},f} \in \mathbb{T}$. This contradicts Lemma 2.4.

If (ii) and (I) hold, $T(\mathbf{1})(0) = T_{\mathbf{1},f}(\mathbf{1})(0) = 0$ and $T(\mathbf{id})(0) = T_{\mathbf{id},f}(\mathbf{id})(0) = 0$. This contradicts Lemma 2.4.

If (ii) and (II) hold, we have $T(f)(0) = T_{\mathbf{id},f}(f)(0) = \lambda_{\mathbf{id},f} f'(0)$ and $(T(f))'(0) = (T_{\mathbf{1},f}(f))'(0) = \mu_{\mathbf{1},f} f(0)$. We also have $T(\mathbf{id})(0) = T_{\mathbf{id},f}(\mathbf{id})(0) = \lambda_{\mathbf{id},f}$ and $(T(\mathbf{1}))'(0) = (T_{\mathbf{1},f}(\mathbf{1}))'(0) = \mu_{\mathbf{1},f}$. Hence we obtain (b). $\square$

*Proof of Theorem 2.1.* Let $T$ be a 2-local isometry on $C^{(2)}$. We note that if Lemma 2.4(i) holds, then Lemma 2.5(a) holds. Suppose that Lemma 2.5(b) holds. Then $T(f)(0) = 0$ for all $f \in C^{(2)}$, which is a contradiction. Similarly, we see that if Lemma 2.4(ii) holds, then Lemma 2.5(b) holds.

By Lemmas 2.3, 2.4 and 2.5, we have

$$\begin{aligned} T(f)(t) &= T(f)(0) + (T(f))'(0)t + (S^2(T(f))'')(t) \\ &= T(\mathbf{1})(0)f(0) + (T(\mathbf{id}))'(0)f'(0)t + (S^2(w(f'' \circ \varphi)))(t) \\ &= \lambda f(0) + \mu f'(0)t + (S^2(w(f'' \circ \varphi)))(t) \end{aligned}$$

or

$$\begin{aligned} T(f)(t) &= T(f)(0) + (T(f))'(0)t + (S^2(T(f))'')(t) \\ &= T(\mathbf{id})(0)f'(0) + (T(\mathbf{1}))'(0)f(0)t + (S^2(w(f'' \circ \varphi)))(t) \\ &= \lambda f'(0) + \mu f(0)t + (S^2(w(f'' \circ \varphi)))(t). \end{aligned}$$

Hence Lemma 2.2 implies that $T$ is a surjective complex-linear isometry on $C^{(2)}$. $\square$

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